THE α-INVARIANT ON $\mathbf{CP^2}\#2\overline{\mathbf{CP^2}}$

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§1. Introduction

The global holomorphic invariant $\alpha_G(M)$ introduced by Tian [6], Tian and Yau [5] is closely related to the existence of Kähler-Einstein metrics. In his solution of the Calabi conjecture, Yau [11] proved the existence of a Kähler-Einstein metric on compact Kähler manifolds with nonpositive first Chern class. Kähler-Einstein metrics do not always exist in the case when the first Chern class is positive, for there are known obstructions such as the Futaki invariant. For a compact Kähler manifold M with positive Chern class, Tian [6] proved that M admits a Kähler-Einstein metric if $\alpha_G(M) > \frac{n}{n+1}$, where $n = \dim M$. In the case of compact complex surfaces, he proved that any compact complex surface with positive first Chern class admits a Kähler-Einstein metric except $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$ [8]. It would be also interesting to find the estimate of the α invariant for $CP^2\#1\overline{CP^2}$ and $CP^2\#2\overline{CP^2}$. In this paper, we apply the Tian-Yau-Zelditch expansion of the Bergman kernel on polarized Kähler metrics to approximate plurisubharmonic functions and compute the α-invariant of $CP^2\#2\overline{CP^2}$. This gives an improvement of Abdesselem's result [1]. More precisely, we shall show that:

Theorem 1 $\alpha_G(CP^2\#2\overline{CP^2})=\frac{1}{3}$.

Let (M,ω) be a compact Kähler manifold, where $\omega=\sqrt{-1}g_{i\bar{j}}dz_i\wedge d\bar{z}_j$. We will also prove Tian's conjecture on the generalized Moser-Trudinger inequality in the special case where $\alpha_G(M) > \frac{n}{n+1}$, for $n = \dim M$. Let

$$P(M,\omega) = \left\{ \phi \mid \omega_\phi = \omega + \partial\overline{\partial}\omega > 0,\ \sup_M \phi = 0 \right\}.$$

Let F_ω and J_ω be the functionals defined on $P(M,\omega)$ by

$$F_\omega(\phi) = J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega - \phi}\omega^n\right)$$

$$J_\omega(M) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \partial\phi \wedge \overline{\partial}\phi \wedge \omega^i \wedge \omega_\phi^{n-i-1}.$$

Assume (M, ω_{KE}) is a Kähler-Einstein manifold with positive first Chern class and $Ric(\omega_{KE}) = \omega_{KE}$, then for any $\phi \in P(M, \omega_{KE})$, Ding and Tian [2] proved the following inequality of Moser-Trudinger tpye:

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}.$$

Tian[9] also conjectured that $\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{(1-\delta)J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}$ for $\delta > 0$ sufficiently small, if ϕ is perpendicular to Λ_1, the space of eigenfunctions of ω_{KE} with eigenvalue one.
We shall prove:

Theorem 2 *Let (M, ω) be a Kähler manifold with positive first Chern class. Assume that $\alpha(M) > \frac{n}{n+1}$, so that M admits a Kähler-Einstein metric ω_{KE}, and there exist constants $\delta = \delta\left(n, \alpha(M)\right)$ and $C = C\left(n, \lambda_2(\omega_{KE}) - 1, \alpha(M)\right)$ such that for any $\phi \in P(M, \omega_{KE})$ which satisfies $\phi \perp \Lambda_1$, we have:*

$$F_{\omega_{KE}}(\phi) \geq \delta J_{\omega_{KE}}(\phi) - C$$

Here $\lambda_2(\omega_{KE})$ is the least eigenvalue of ω_{KE} which is bigger than 1.

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§2. Holomorphic approximation of psh

In this section, we will employ the technique in [7, 12] to obtain the approximation of plurisubharmonic functions by logarithms of holomorphic sections of line bundles. The Tian-Yau-Zelditch asymptotic expansion of the potential of the Bergman metric is given by the following theorem [7, 12].

Theorem 2.1 *Let M be a compact complex manifold of dimension n and let $(L,h) \to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$. For each $m \in N$, h induces a Hermitian metric h_m on L^m. Let $\{S_0^m, S_1^m, ..., S_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, $d_m = \dim H^0(M, L^m)$, with respect to the inner product:*

$$(S_1, S_2)_{h_m} = \int_M h_m(S_1(x), S_2(x)) dV_g,$$

where $dV_g = \frac{1}{n!}\omega_g^n$ is the volume form of g. Then there is a complete asymptotic expansion:

$$\sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 = a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + ...$$

for some smooth coefficients $a_j(x)$ with $a_0 = 1$. More precisely, for any k:

$$\left\| \sum_{i=0}^{d_m-1} ||S_i^m(x)||_{h_m}^2 - \sum_{j<R} a_j(x) m^{n-j} \right\|_{C^k} \leq C_{R,k} m^{n-R}$$

where $C_{R,k}$ depends on R, k and the manifold M.

Let

$$\begin{aligned} \widetilde{\omega}_g &= \omega_g + \partial\overline{\partial}\phi > 0 \\ \widetilde{h} &= h e^{-\phi} \end{aligned}$$

Let $\widetilde{h}_m$ be the induced Hermitian metric of $\widetilde{h}$ on L^m, $\{\widetilde{S}_0^m, \widetilde{S}_{1,...,}^m \widetilde{S}_{d_m-1}^m\}$ be any orthonormal basis of $H^0(M, L^m)$, where $d_m = \dim H^0(M, L^m)$, with respect to the inner product

$$(S_1, S_2)_{\widetilde{h}_m} = \int_M \widetilde{h}_m(S_1(x), S_2(x)) dV_{\widetilde{g}}\,.$$

By Theorem 2.1, we have

$$\begin{aligned} \sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||_{\widetilde{h}_m}^2 &= \widetilde{a}_0(x)m^n + \widetilde{a}_1(x)m^{n-1} + \widetilde{a}_2(x)m^{n-2} + ... \\ &= \left(\sum_{i=0}^{d_m-1} ||\widetilde{S}_i^m(x)||_{h_m}^2 \right) e^{-m\phi}. \end{aligned}$$

Thus

$$\phi = \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1}||\widetilde{S}_i^m(x)||^2_{\widetilde{h}_m}\right) - \frac{1}{m}\log\left(\widetilde{a}_0(x)m^n + \widetilde{a}_1(x)m^{n-1} + \widetilde{a}_2(x)m^{n-2} + ...\right)$$

As $m \to +\infty$, we obtain

$$\begin{aligned} & \frac{1}{m}\log\left(\widetilde{a}_0(x)m^n + \widetilde{a}_1(x)m^{n-1} + \widetilde{a}_2(x)m^{n-2} + ...\right) \\ = \;& \frac{1}{m}\log m^n(1 + \widetilde{a}_1(x)m^{-1} + \widetilde{a}_2(x)m^{-2} + ...) \\ = \;& \frac{n}{m}\log m + \frac{1}{m}\log(1 + O(\frac{1}{m})) \to 0 \end{aligned}$$

Thus we have the following corollary of the Tian-Yau-Zelditch expansion.

Corollary 2.1

$$\left\|\phi - \frac{1}{m}\log\left(\sum_{i=0}^{d_m-1}||\widetilde{S}_i^m(x)||^2_{h_m}\right)\right\|_{C^k} \to 0, \; as\; m \to +\infty.$$

In other words, any plurisubharmonic function can be approximated by the logarithms of holomorphic sections of L^m.

§3. Proof of Theorem 1

Let M be the blow-up of CP^2 at two points and π its natural projection. Without loss of generality, we may assume the two points are $p_1 = [0,1,0]$ and $p_2 = [0,0,1]$. Then M is a subvariety of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$Z_0X_1 = Z_1X_0, \;\; Z_0Y_2 = Z_2Y_0$$

where Z_i, X_j, Y_k are respectively the homogeneous coordinates on CP^2, CP^1 and CP^1.

Let G be the automorphism group acting on $CP^2 \times CP^1 \times CP^1$ generated by θ_j and permutations τ $(0 \leq i \leq 2)$

$$\theta_j : [Z_0, Z_j, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_je^{i\theta}, Z_2] \times [X_0, X_1] \times [Y_0, Y_2]$$

for $\theta \in [0, 2\pi)$, and

$$\tau : [Z_0, Z_1, Z_2] \times [X_0, X_1] \times [Y_0, Y_2] \to [Z_0, Z_2, Z_1] \times [Y_0, Y_2] \times [X_0, X_1] .$$

Let π_0, π_1, π_2 be the projection from $CP^2 \times CP^1 \times CP^1$ onto CP^2, CP^1 and CP^1. Define ω by

$$\begin{aligned}\omega &= \pi_0^*\omega_0 + \pi_1^*\omega_1 + \pi_2^*\omega_2 \\ &= \partial\overline{\partial}\log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) + \partial\overline{\partial}\log(|X_0|^2 + |X_1|^2) + \\ &\quad \partial\overline{\partial}\log(|Y_0|^2 + |Y_2|^2)\end{aligned}$$

where ω_0, ω_1, ω_2 are the Fubini-Study metrics CP^2, CP^1 and CP^1. By explicit calculation, it can be shown that $\omega|_M$ is in the first Chern class of M(see [1]).

Consider the divisor

$$\{[0, Z_1, Z_2] \times CP^1 \times CP^1\} + \{CP^2 \times [1, 0] \times CP^1\} + \{CP^2 \times CP^1 \times [1, 0]\}$$

which defines a line bundle (L, h) on $CP^2 \times CP^1 \times CP^1$, where

$$h = \frac{1}{(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)(|X_0|^2 + |X_1|^2)(|Y_0|^2 + |Y_2|^2)},$$

then $(L, h)|_M \to M$ defines the anticanonical line bundle on M whose curvature form $-\partial\overline{\partial}\log h$ gives the first Chern class on M.
Since $M \backslash \{\pi^{-1}\{p_1\} \cup \pi^{-1}\{p_2\}\}$ is isomorphic to $CP^2 \backslash \{p_1, p_2\}$, if we choose the inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ on CP^2, the Kähler metric

$$\omega_{g_0} = \partial\overline{\partial}\log(1 + |z_1|^2 + |z_2|^2) + \partial\overline{\partial}\log(1 + |z_1|^2) + \partial\overline{\partial}\log(1 + |z_2|^2)$$

can be extended to a Kähler metric g_0 on M which belongs to $c_1(M)$. If we take different inhomogeneous coordinates $(w_0, w_1) = [w_0, w_1, 1]$, the corresponding Kähler metric is

$$\omega_{g_1} = \partial\overline{\partial}\log(1 + |w_0|^2 + |w_1|^2) + \partial\overline{\partial}\log(1 + |w_0|^2) + \partial\overline{\partial}\log(|w_0|^2 + |w_1|^2)$$

and we have

$$
\begin{aligned}
\det g_0 &= \frac{1}{(1+|z_1|^2+|z_2|^2)^3} + \frac{1}{(1+|z_1|^2+|z_2|^2)^2(1+|z_1|^2)} \\
&\quad + \frac{1}{(1+|z_1|^2+|z_2|^2)^2(1+|z_2|^2)} + \frac{1}{(1+|z_1|^2)^2(1+|z_2|^2)^2} \\
\det g_1 &= \frac{1}{(1+|w_0|^2+|w_1|^2)^3} + \frac{1}{(1+|w_0|^2+|w_1|^2)^2(|w_0|^2+|w_1|^2)} \\
&= \frac{1}{(1+|w_0|^2+|w_1|^2)^2(1+|w_0|^2)} + \frac{|w_0|^2}{(1+|w_0|^2)^2(|w_0|^2+|w_1|^2)^2}.
\end{aligned}
$$

Consider the line bundle $(L^N, h_N) \to CP^2 \times CP^1 \times CP^1$. Then

$$
\dim H^0(CP^2 \times CP^1 \times CP^1, O(L^N)) = \frac{(N+1)^3(N+2)}{2}
$$

and $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_2}\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N,}$ is an orthogonal basis for $H^0(CP^2 \times CP^1 \times CP^1, O(L^N))$.
Let M_1 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$
Z_0 X_1 = Z_1 X_0
$$

and M_2 be the hypersurface of $CP^2 \times CP^1 \times CP^1$ defined by the equations

$$
Z_0 Y_2 = Z_2 Y_0.
$$

Then $M = M_1 \cap M_2$.
In view of the short exact sequences

$$
\begin{aligned}
0 &\to O(L^N - [M_1]) \to O(L^N) \to O(L^N|_{M_1}) \to 0 \\
0 &\to O(L^N|_{M_1} - [M]) \to O(L^N|_{M_1}) \to O(L^N|_M) \to 0
\end{aligned}
$$

we can choose N sufficiently large so that

$$
H^1(CP^2 \times CP^1 \times CP^1, O(L^N - [M_1])) = H^1(M_1, O(L^N|_{M_1} - [M])) = 0.
$$

Then $H^0(CP^2 \times CP^1 \times CP^1, O(L^N)) \to H^0(M_1, O(L^N|_{M_1})) \to 0$

$$
H^0(M_1, O(L^N|_{M_1})) \to H^0(M, O(L^N|_M)) \to 0
$$

Hence $\{Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}|_M\}_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N}$ contains an orthogonal basis for $H^0(M, O(L^N|_M))$ and

$$||Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} X_0^{j_0} X_1^{j_1} Y_0^{k_0} Y_2^{k_1}||_{h_N}^2 = \frac{\left|Z_0^{i_0} Z_1^{i_1} Z_2^{i_2} Z_0^{j_0} Z_1^{j_1} Z_0^{k_0} Z_2^{k_2}\right|^2}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(Z_0|^2+|Z_1|^2)(Z_0|^2+|Z_2|^2))^N}$$

on $CP^2\backslash\{p_1, p_2\}$. By Corollary 2.1, for any φ in $P_G(M, \omega_g)$, we have on $CP^2\backslash\{p_1, p_2\}$,

$$\begin{aligned} & \varphi([Z_0, Z_1, Z_2]) \\ = \quad & \lim_{N\to\infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} \end{aligned}$$

for some coefficients $a^{(N)}_{(\varphi)i_0 i_1 i_2 j_0 j_1 k_0 k_2} = a^{(N)}_{(\varphi)i_0 i_2 i_1 k_0 k_2 j_0 j_1}$ which is set in the view of the group action by G.

Lemma 3.1 $\frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} \leq Const.$

Proof On the patch $U_0 = \{Z_0 \neq 0\}$, let $z_1 = \frac{Z_1}{Z_0}$ and $z_2 = \frac{Z_2}{Z_0}$,

$$\begin{aligned} & \frac{1}{n} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} \\ \leq \quad & \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{(1+|z_1|^2+|z_2|^2)^n (1+|z_1|^2)^n (1+|z_2|^2)^n} \right) \\ \leq \quad & \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2}{1+|z_1^{i_1+j_1} z_2^{i_2+k_2}|^2} \right) \\ \leq \quad & \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} 1 \right) \\ = \quad & \frac{1}{n} \log \frac{(n+1)^3(n+2)}{2} \end{aligned}$$

On the patch $U_2 = \{Z_2 \neq 0\}$, let $w_0 = \frac{Z_0}{Z_2}$ and $w_1 = \frac{Z_1}{Z_2}$,

$$
\begin{aligned}
& \frac{1}{n} \log \frac{\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} \\
\leq\ & \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|w_0^{i_0+j_0+k_0} w_1^{i_1+j_1}|^2}{(1+|w_0|^2+|w_1|^2)^n (1+|w_0|^2)^n (|w_0|^2+|w_1|^2)^n} \right) \\
\leq\ & \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} \frac{|w_0^{i_0+j_0+k_0} w_1^{i_1+j_1}|^2}{|w_0^{i_0+j_0+k_0} w_1^{i_1+j_1}|^2} \right) < \frac{1}{n} \log \left(\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} 1 \right) \\
=\ & \frac{1}{n} \log \frac{(n+1)^3(n+2)}{2}
\end{aligned}
$$

This inequality holds for the patch $U_1 = \{Z_1 \neq 0\}$, and so the lemma is proved.

Lemma 3.2 *There exists $\varepsilon > 0$ such that for any $\varphi \in P_G(M, \omega_g)$ and N, there exist $n > N$, $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ with $i_0+i_1+i_2 = j_0+j_1 = k_0+k_2 = n$, and $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} > \varepsilon$.*

Proof Otherwise, for any $\varepsilon > 0$, there exists φ and N, such that for any $n > N$ and any $i_0, i_1, i_2, j_0, j_1, k_0, k_2$ satisfying $i_0+i_1+i_2=j_0+j_1=k_0+k_2=n$, we have $(a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2})^{\frac{1}{n}} < \varepsilon$. By choosing n large enough and with the lemma above, we have

$$
\begin{aligned}
& \varphi([Z_0, Z_1, Z_2]) \\
\leq\ & \frac{1}{n} \log \frac{\max |a^{(n)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2}|^2 \sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} + \varepsilon \\
\leq\ & \frac{1}{n} \log \frac{\sum\limits_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=n} |Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^n} + 2\log\varepsilon + \varepsilon \\
\leq\ & \log\varepsilon + const
\end{aligned}
$$

Since ε could be arbitrarily small, the above inequality would imply that $\varphi \to -\infty$ uniformly, which contradicts the fact that $\sup_M \varphi = 0$.

Proof of the theorem:

$$
\begin{aligned}
&\varphi([Z_0, Z_1, Z_2]) \\
= \;& \lim_{N\to\infty} \frac{1}{N} \log \frac{\sum_{i_0+i_1+i_2=j_0+j_1=k_0+k_2=N} |a^{(N)}_{(\varphi)i_0i_1i_2j_0j_1k_0k_2} Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} \\
\geq \;& \frac{1}{N} \log \frac{|Z_0^{i_0+j_0+k_0} Z_1^{i_1+j_1} Z_2^{i_2+k_2}|^2 + |Z_0^{i_0+j_0+k_0} Z_1^{i_2+k_2} Z_2^{i_1+j_1}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} + C_1 \\
\geq \;& \frac{1}{N} \log \frac{|{Z_0}^m Z_1^{\frac{3}{2}N-\frac{m}{2}} Z_2^{\frac{3}{2}N-\frac{m}{2}}|^2}{((|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2))^N} + C_1 \\
\geq \;& \log \frac{|{Z_0}^{\frac{2m}{N}} Z_1^{3-\frac{m}{N}} Z_2^{3-\frac{m}{N}}|^2}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)} + C_1
\end{aligned}
$$

where $i_0 + j_0 + k_0 = m, i_1 + j_1 + i_2 + k_2 = 3N - m$.

On the patch $U_0 = \{Z_0 \neq 0\}$,

$$
\begin{aligned}
&\int_{U_0\cap\{0<|z_1|,|z_2|<1\}} e^{-\alpha\varphi}\omega_{g_0}^2 \\
\leq \;& C_1 \int_{0<|z_1|,|z_2|<1} e^{-\alpha \log \frac{|Z_0|^{\frac{2m}{N}} |Z_1|^{3-\frac{m}{N}} |Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}} \omega_{g_0}^2 \\
= \;& C_1 \int_{0<|z_1|,|z_2|<1} \frac{(|Z_0|^2+|Z_1|^2+|Z_2|^2)^\alpha (|Z_0|^2+|Z_1|^2)^\alpha (|Z_0|^2+|Z_2|^2)^\alpha}{|Z_0|^{\frac{2\alpha m}{N}} |Z_1|^{3\alpha-\frac{\alpha m}{N}} |Z_2|^{3\alpha-\frac{\alpha m}{N}}} \omega_{g_0}^2 \\
\leq \;& C_2 \int_{0<|z_1|,|z_2|<1} \frac{(1+|z_1|^2+|z_2|^2)^\alpha (1+|z_1|^2)^\alpha (|1+|z_2|^2)^\alpha}{|z_1|^{3\alpha-\frac{\alpha m}{N}} |z_2|^{3\alpha-\frac{\alpha m}{N}}} dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \\
\leq \;& C_3 \int_{0<|z_1|,|z_2|<1} \frac{1}{|z_1|^{3\alpha-\frac{\alpha m}{N}} |z_2|^{3\alpha-\frac{\alpha m}{N}}} dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2 \\
\leq \;& C_3 \int_{0<|z_1|,|z_2|<1} \frac{1}{|z_1|^{3\alpha} |z_2|^{3\alpha}} dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2
\end{aligned}
$$

On the patch $U_2 = \{Z_2 \neq 0\}$,

$$
\begin{aligned}
&\int_{U_1\cap 0<|w_0|,|w_1|\leq 1} e^{-\alpha\varphi}\omega_{g_1}^2 \\
\leq \;& C_4 \int_{0<|w_0|,|w_1|\leq 1} e^{-\alpha \log \frac{|Z_0|^{\frac{2m}{N}} |Z_1|^{3-\frac{m}{N}} |Z_2|^{3-\frac{m}{N}}}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}} \omega_{g_1}^2
\end{aligned}
$$

$$
\begin{aligned}
&= C_4\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}}\omega_{g_1}^2\\
&\leq C_5\int_{0<|w_0|,|w_1|\leq 1}\frac{(1+|w_0|^2+|w_1|^2)^\alpha(1+|w_0|^2)^\alpha(|w_0|^2+|w_1|^2)^\alpha}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)}dw_0\wedge d\overline{w}_0\wedge dw_1\wedge d\overline{w}_1\\
&\leq C_6\int_{0<|w_0|,|w_1|\leq 1}\frac{1}{|w_0|^{\frac{2\alpha m}{N}}|w_1|^{3\alpha-\frac{\alpha m}{N}}(|w_0|^2+|w_1|^2)^{1-\alpha}}dw_0\wedge d\overline{w}_0\wedge dw_1\wedge d\overline{w}_1\\
&\leq C_6\int_{t=0}^1\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}(s+t)^{1-\alpha}}dsdt\\
&\leq C_6\int_{s=0}^1\frac{1}{s^{\frac{\alpha m}{N}}t^{\frac{3}{2}\alpha-\frac{\alpha m}{2N}}s^{(1-\alpha)p}t^{(1-\alpha)q}}dsdt
\end{aligned}
$$

where $(p+q=1)$.

Case 1: If $1\leq\frac{m}{N}\leq 3$, then

$$
\begin{aligned}
\frac{\alpha m}{N}+(1-\alpha)p &< 1\Leftarrow\alpha<\frac{1-p}{3-p}<\frac{1-p}{\frac{m}{N}-p}\\
3\alpha-1 &< 1\\
\frac{3}{2}\alpha-\frac{\alpha m}{2N}+(1-\alpha)q &< 1\Leftarrow\alpha<1<\frac{1-q}{\frac{3}{2}-\frac{m}{2N}-q}
\end{aligned}
$$

Case 2: If $0<\frac{m}{N}<1$, then

$$
\begin{aligned}
\frac{\alpha m}{N}+(1-\alpha)p &< 1\Leftarrow\alpha<1\\
3\alpha-1 &< 1\\
\frac{3}{2}\alpha-\frac{\alpha m}{2N}+(1-\alpha)q &< 1\Leftarrow\alpha<\frac{1-q}{\frac{3}{2}-q}
\end{aligned}
$$

So we could choose any $\alpha<\frac{1}{3}$, which implies that $\alpha_G(M,\omega)\geq\frac{1}{3}$.

Conversely, we choose

$$
\begin{aligned}
\varphi_\varepsilon &= \log\left(\frac{|Z_0|^6}{(|Z_0|^2+|Z_1|^2+|Z_2|^2)(|Z_0|^2+|Z_1|^2)(|Z_0|^2+|Z_2|^2)}+\varepsilon\right)\\
&\quad -\log(1+\varepsilon)\\
&\in P_G(M,\omega)
\end{aligned}
$$

Then we have $\sup_M \varphi_\varepsilon = 0$ and $\varphi_\varepsilon = \log \frac{\varepsilon}{1+\varepsilon}$ on the exceptional divisors. Furthermore, we have

$$\lim_{\varepsilon \to 0} \int_M e^{-\alpha \varphi_\varepsilon} \omega^2 = \infty, \ for\ any\ \alpha > \frac{1}{3}.$$

Hence we have shown $\alpha_G(M, \omega) = \frac{1}{3}$.

§4. Proof of Theorem 2

In this section, we will prove the generalized Moser-Trudinger inequality on any Kähler manifold M of dimension n whose $\alpha(M)$ is greater than $\frac{n}{n+1}$. The following theorem is due to Tian and Zhu [10].

Theorem 4.1 *Let (M, ω) be a Kähler-Einstein manifold with $Ric(\omega) = \omega$, then there exist constants $\delta = \delta(n)$ and $C = C(n, \lambda_2(\omega) - 1) \geq 0$ such that for any $\phi \in P(M, \omega)$ which satisfies $\phi \perp \Lambda_1$, we have*

$$F_\omega(\phi) \geq J_\omega(\phi)^\delta - C,$$

which is the same as

$$\frac{1}{V} \int_M e^{-\phi} \omega^n \leq C e^{J_\omega(\phi) - \frac{1}{V} \int_M \phi \omega^n - J_\omega(\phi)^\delta}.$$

This implies in particular the Moser-Trudinger inequality on S^2, which reads

$$\frac{1}{4\pi} \int_{S^2} e^{-\phi} \omega \leq e^{\frac{1}{8\pi} \int_{S^2} |\nabla \phi|^2 \omega - \frac{1}{4\pi} \int_{S^2} \phi}$$

For any $\phi \in P(M, \omega)$, put $\omega' = \omega_\phi = \omega + \partial\overline{\partial}\phi$ and $Ric(\omega) = \omega + \partial\overline{\partial} h_\omega$. Consider the Monge-Ampère equation

$$(\omega' + \partial\overline{\partial}\psi)^n = e^{h_\omega - t\psi} \omega'^n$$

We will use the continuity method backwards and let ϕ_t be a smooth family which solve the above equation.

The following lemmas are well-known [9], but we add the proofs for the sake of completeness.

Lemma 4.1 *$Ric(\omega_t) \geq t\omega_t$, and we have equality if and only if $t = 1$.*

Proof

$$\begin{aligned} Ric(\omega_t) &= -\partial\overline{\partial}\log\omega_t^n = -\partial\overline{\partial}\log\frac{\omega_t^n}{\omega^n} + Ric(\omega) = -\partial\overline{\partial}\,(h_\omega - t\phi_t) + \omega + \partial\overline{\partial}h_\omega \\ &= \omega + t\phi_t = t\omega_t + (1-t)\omega \geq t\omega_t. \end{aligned}$$

Lemma 4.2 *For any $\phi \in P(M,\omega)$, if the Green's function of $\omega' = \omega + \partial\overline{\partial}\phi$ is bounded from below, we have:*

$$-\inf_M \phi \leq \frac{1}{V}\int_M (-\phi)\omega'^n + C.$$

Proof Since $\omega + \partial\overline{\partial}\phi = \omega'$ and $\omega' - \partial\overline{\partial}\phi > 0$, we have $\Delta_{\omega'}\phi \leq n$.

$$\begin{aligned} -\phi &= \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M \Delta_{\omega'}\phi(y)G_{\omega'}(x,y)\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + \frac{1}{V}\int_M n(G_{\omega'}(x,y) - \inf G_{\omega'}(x,y))\omega'^n \\ &\leq \frac{1}{V}\int_M (-\phi)\omega'^n + C. \end{aligned}$$

Let (M,ω) be a Kähler-Einstein manifold with $\mathrm{Ric}(\omega) = \omega$ and let $P(M,\omega,K)$ $= \{\phi \in P(M,\omega) \mid G_{\omega+\partial\overline{\partial}\phi}(x,y) \geq -K)$. Then we have:

Proposition 4.1 *Let (M,ω) be a Kähler-Einstein manifold with $Ric\,(\omega) = \omega$. If $\alpha(M) > \frac{n}{n+1}$, then there exist constants $\delta(n,\alpha,K)$ and $C(n,\alpha,\lambda_2(\omega)-1,K)$ such that for any $\phi \in P(M,\omega,K)$, we have*

$$F_\omega(\phi) \geq \delta J_\omega(\phi) - C.$$

Proof Let $\omega' = \omega + \partial\overline{\partial}\phi$, where $\phi \in P(M,\omega,K)$.

$$\begin{aligned} \frac{1}{V}\int_M e^{-\alpha\phi}\omega^n &= \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2+\varepsilon)\phi}\omega^n \\ &\leq \frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n e^{-\varepsilon\inf_M\phi}, \end{aligned}$$

take $p = \frac{1}{\alpha_1}, q = \frac{1}{1-\alpha_1}$, we have

$$
\begin{aligned}
\frac{1}{V}\int_M e^{-(\alpha_1+\alpha_2)\phi}\omega^n &\le \frac{1}{V}(\int_M e^{-\alpha_1 p\phi}\omega^n)^{1/p}(\int_M e^{-\alpha_2 q\phi}\omega^n)^{1/q}\\
&= \frac{1}{V}(\int_M e^{-\phi}\omega^n)^{\alpha_1}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1}\\
&\le Ce^{\alpha_1 J_\omega(\phi)-\frac{\alpha_1}{V}\int_M \phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{1-\alpha_1}
\end{aligned}
$$

by Lemma 4.2,

$$
\begin{aligned}
e^{-\varepsilon \inf_M \phi} &\le e^{\frac{\varepsilon}{V}\int_M(-\phi)\omega'^n+C}\\
&= e^{\varepsilon I_\omega(\phi)-\frac{\varepsilon}{V}\int_M \phi\omega^n+C}\\
&\le e^{\varepsilon(n+1)J_\omega(\phi)-\frac{\varepsilon}{V}\int_M \phi\omega^n+C}.
\end{aligned}
$$

By Holder inequality,

$$
\begin{aligned}
\frac{1}{V}\int_M e^{-\phi}\omega^n &\le (\frac{1}{V}\int_M e^{-\alpha\phi}\omega^n)^{\frac{1}{\alpha}}\\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{\alpha_1+\varepsilon}{\alpha V}\int_M\phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}\phi}\omega^n)^{\frac{1-\alpha_1}{\alpha}}\\
&= Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n+\frac{\alpha_2}{V}\int_M(\phi-\sup\phi)\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}}\\
&\le Ce^{\frac{\alpha_1+(n+1)\varepsilon}{\alpha}J_\omega(\phi)-\frac{1}{V}\int_M\phi\omega^n}(\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi-\sup\phi)}\omega^n)^{\frac{1-\alpha_1}{\alpha}}
\end{aligned}
$$

We need to determine α_1, α_2, ε, which satisfy the following conditions

$$
\begin{aligned}
\alpha &= \alpha_1+\alpha_2+\varepsilon > 1\\
\alpha &> \alpha_1+(n+1)\varepsilon\\
1 &> \alpha_1
\end{aligned}
$$

So we will choose

$$
\begin{aligned}
\alpha_2 &= n\varepsilon+\varepsilon'\\
\alpha_1 &= 1-\alpha_2-\varepsilon+\varepsilon" = 1-(n+1)\varepsilon-\varepsilon'+\varepsilon"
\end{aligned}
$$

where ε , ε' , $\varepsilon" << 1$, and $\varepsilon' = o(\varepsilon)$, $\varepsilon" = o(\varepsilon')$.
Since $\alpha(M) > \frac{n}{n+1}$, then we can choose ε, ε' , $\varepsilon"$ small enough, then we have

$$
\frac{\alpha_2}{1-\alpha_1} = \frac{n\varepsilon+\varepsilon'}{(n+1)\varepsilon+\varepsilon'-\varepsilon"} < \alpha(M)
$$

and

$$\int_M e^{-\frac{\alpha_2}{1-\alpha_1}(\phi - \sup \phi)} \omega^n < Const.$$

Combined with the inequalities above, we have

$$\frac{1}{V}\int_M e^{-\phi}\omega^n \leq Ce^{(1-\delta)J_\omega(\phi) - \frac{1}{V}\int_M \phi\omega^n}$$

Which proves the lemma.

Proof of Theorem 2

We assume ω is the Kähler-Einstein metric of M. For any $\phi \in P(M,\omega)$, put $\omega' = \omega + \partial\overline{\partial}\phi$. Consider $(\omega' + \partial\overline{\partial}\psi) = e^{h_{\omega'} + t\psi}$. By solving the Monge-Ampère equation backwards, we get the solutions ϕ_t, and $\phi_1 = -\phi$.

For $t > \frac{1}{2}$, let $\omega_t = \omega' + \partial\overline{\partial}\phi_t = \omega + \partial\overline{\partial}(\phi_t - \phi_1)$, by Lemma 4.1,

$$Ric(\omega_t) \geq \frac{1}{2}\omega_t.$$

which shows that the Green function of ω_t is uniformly bounded from below.

Thus by proposition 4.1 and the calculation in [10] we have

$$\begin{aligned} F_\omega(\phi_t - \phi_1) &\geq \delta J_\omega(\phi_t - \phi_1) - C \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2 \end{aligned}$$

and consequently,

$$\begin{aligned} n(1-t)J_\omega(\phi) &= n(1-t)J_{\omega'}(\phi_1) \\ &\geq (1-t)(I_{\omega'}(\phi_1) - J_{\omega'}(\phi_1)) \\ &\geq F_{\omega'}(\phi_t) - F_{\omega'}(\phi_1) \\ &= F_\omega(\phi_t - \phi_1) \\ &\geq C_1 osc_M(\phi_t - \phi_1) - C_2 \end{aligned}$$

$$\begin{aligned} F_\omega(\phi) &= -F_{\omega'}(-\phi) \\ &= \int_0^1 (I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t))dt \\ &\geq (1-t)(I_{\omega'}(\phi_t) - J_{\omega'}(\phi_t)) \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1-t}{n} J_{\omega'}(\phi_t) \\
&\geq \frac{1-t}{n} J_{\omega'}(\phi_1) - 2(1-t)(C_1 osc_M(\phi_t - \phi_1) - C_2) \\
&\geq \frac{1-t}{n} J_{\omega}(\phi) - 2(1-t)^2 n C_1 J_{\omega}(\phi) - C_3
\end{aligned}
$$

The theorem follows by choosing $(1-t) < \frac{1}{2n^2C_1}$.

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